Logicist responses to Kant: (early) Frege and (early) Russell[1]

The "textbook" account of the movement in the philosophy of mathematics called "logicism" is summed up in the title of chapter IX of Raymond Wilder's Introduction to the Foundations of Mathematics: "The Frege-Russell Thesis: Mathematics an Extension of Logic." (Wilder 1965, 219.) A. D. Irvine spells out the view implicit in this title in the (on-line) Stanford Encyclopedia of Philosophy entry for "Principia Mathematica" (Irvine 1996):

> Logicism is the view that (some or all of) mathematics can be reduced to (formal) logic ... In its essentials, this thesis was first advocated in the late 17th century by Gottfried Leibniz. Later, the idea was defended in much greater detail by Gottlob Frege. ... it was not until 1879, when Frege developed the necessary logical apparatus, that the project of logicism could be said to have become technically viable. Following another five years' work, Frege arrived at the definitions necessary for logicising arithmetic. During the 1890s he worked on many of the essential derivations. However, with the discovery of paradoxes such as Russell's paradox at the turn of the century, it appeared that additional resources would be required if logicism were to succeed. By 1903, both Whitehead and Russell had come to the same conclusion. By this time, both men were also in the initial stages of preparing second volumes to earlier books on related topics: Whitehead's 1898 A Treatise on Universal Algebra and Russell's 1903 The Principles of Mathematics. Since their research overlapped considerably, they began collaboration on what was eventually to become Principia Mathematica.

According to this kind of story, Russell (and Whitehead) and Frege were engaged in a common project, called "logicism," the defense of the view that "(some or all of) mathematics can be reduced to (formal) logic;" this view, and the project of defending it, underwent development but remained essentially and recognizably the same from 1879 to 1910. Howard Stein, in the entry for "logicism" in the Routledge Encyclopedia of Philosophy, sounds a note of caution about such textbook tales of the origin of logicism, however (Stein 1998):

> Although the thesis that arithmetic is a part of logic was stated quite explicitly by Frege

> ... and by Russell ..., it is surprisingly difficult to determine the exact content of this thesis; and, indeed, to determine whether it should be understood to mean the same thing to each of them. For the obvious question presents itself: exactly what are we to understand by “logic”?

Moreover, Stein points out, on the epistemological question of the grounding of our knowledge of logic itself,

> Frege and Russell appear to disagree sharply. Both regard the thesis as being in opposition to Kant; but whereas Frege, in denying that arithmetic is based upon any Kantian “pure intuition” and maintaining that it is entirely grounded in logic, concludes that Kant was wrong to consider arithmetical propositions synthetic, and holds by contrast that, as propositions of logic, they are analytic, ... Russell denies the latter, and holds that logic itself is synthetic in character.

Stein concludes:

> Yet even here the issue is not so clearly drawn; for one must still ask whether Frege, in affirming that logic is analytic and not synthetic, and Russell, in affirming the opposite, understood the words “analytic” and “synthetic” in the same way.

In this paper I aim to pursue these questions of Stein’s, focusing on the versions of logicism presented in Frege’s and Russell’s early works, The Foundations of Arithmetic (1884), and The Principles of Mathematics (1903). I will argue that, in these works, Frege and Russell’s conceptions of analyticity, syntheticity, and logic itself, are sufficiently different to put into question the idea that they are really engaged in a common project which it is useful to put under one heading, “logicism.”[2]

To begin with, Frege recognizes and accepts the importance of Kant’s threefold distinction of truths into analytic, synthetic a posteriori, and synthetic a priori; and he accepts Kant’s classification of geometry as synthetic a priori. Thus Frege’s logicism extends to arithmetic (including under this heading real analysis) but not to geometry; Frege does not hold that all of mathematics is reducible to logic (hence his title, The Foundations of ARITHMETIC). On the other hand, as we will see, Russell rejects the intelligibility of any contentful analytic/synthetic distinction, other than that between bare identities (A is A) and other truths. He therefore treats geometry as not interestingly distinct from arithmetic in this respect, and maintains that all of mathematics is reducible to logic (hence his title, The Principles of MATHEMATICS). Thus Frege and Russell's different reactions to Kant reveal very different attitudes to the question Kant posed about the synthetic a priori status of mathematics. To clarify all this, I will first discuss the treatment of the analytic/synthetic distinction in Kant, Frege, and Russell, and the application of this distinction to logic and mathematics. I will then take up the differing views of conceptual analysis, and of the role of logic therein, held by Frege and Russell. I will argue that (early) Frege and (early) Russell have different underlying conceptions of logic, so that, as Stein suggests, their "logicisms" are not really versions of the same project.[3]

1. Kant on the analytic/synthetic distinction[4]

Kant famously argues in the first Critique and in the Prolegomena that mathematics is synthetic a priori, and claims in the Prolegomena that if Hume had recognized the synthetic a priori status of mathematics, he would not have been so skeptical of the possibility of metaphysics, which must have the same status. (CPR, B14-B17; PFM, 18-20.) Yet Kant himself appears to draw the crucial distinction between analytic and synthetic judgments in at least three ways, which do not obviously cohere with one another:

(A) the concept-containment distinction: in an analytic judgment, the subject concept contains the predicate concept; in a synthetic judgment, the subject concept is linked to the predicate concept, but not by containing it. In the Critique of Pure Reason, this is given as the official explanation and (B) and (C) come in as further explanations. (CPR, A6/B10-A10/B14.)

(B) the explicative/ampliative distinction: the analytic/synthetic distinction concerns the content of our judgments, and not their origin. Synthetic judgments are ampliative and extend our knowledge; analytic judgments are merely explicative and do nothing but make explicit conceptual connections, which are already implicit in our grasp of concepts. In the Prolegomena, the distinction is introduced in this way, and (A) and (C) are then brought in as further explanations. (PFM, 16-18.)

(C) a distinction between the principles of analytic and synthetic judgments: both the Critique and the Prolegomena tell us that analytic judgments are based on the principle of contradiction, whereas synthetic judgments require some other principle, which the Critique eventually reveals to be that "every object stands under the necessary conditions of the synthetic unity of the manifold of intuition in a possible experience." (CPR, A158/B197.) In the Logic, the analytic/synthetic distinction is introduced in similar terms: "Propositions whose certainty rests on identity of concepts (of the predicate with the notion of the subject) are called analytic propositions. Propositions whose truth is not grounded on identity of concepts must be called synthetic." (LL, 606.) The subsequent discussion mentions neither (A) nor (B) explicitly but provides an example suggestive of (A) and a distinction similar to (though interestingly different from) (B) (LL, 606-607):

> An example of an analytic proposition is, To everything x, to which the concept of body (a + b) belongs, belongs also extension (b). An example of a synthetic proposition is, To everything x, to which the concept of body (a + b) belongs, belongs also attraction (c). Synthetic propositions increase cognition materialiter [materially], analytic ones merely formaliter [formally]. (LL, 606-607.)

Kant seems to have thought of these three distinctions (A)-(C) as interchangeable; when he employs one of them to explain the analytic/synthetic distinction he typically introduces the other two as if they were consequences of his explanation. In order to appreciate the unity of these three distinctions in Kant's thought, it will be useful to briefly review some relevant points from his Logic.

In the Logic, Kant distinguishes between concepts, as general representations, and intuitions, as singular representations. Concepts are arranged in a hierarchy of "containment." For example, the concept "animal" is contained in the concept "mammal" which is in turn contained in the concept "whale." (LL, 593-595.) We can understand containment here as a matter of necessary implication (what is implied is what is "folded into" a given concept, so what is contained in it); we can also understand containment in terms of the metaphor of analysis of a whole into its component parts, where the parts are linked logically by conjunction (so whale = mammal + animal + ... as in the passage from the Logic quoted above). The content of a given concept is comprised of all the concepts contained in the given concept. (LL, 593.)

Kant applies this notion of content not only to concepts, but also to cognitions in general. Thus, he speaks of the content of a cognition as a matter of its "richness or ... logical importance and fruitfulness... as ground of many and great consequences," and he speaks of the extension of a cognition as its "multitude and manifoldness." (LL, 549-550.) If we apply this notion to judgments, in particular, we will find that the content of a judgment is a matter of its consequences, of what follows from it.

Kant's account of judgment itself is "the representation of the unity of the consciousness of various representations" (LL, 597); judgment links representations together and represents them as in some sense belonging together. Kant's paradigm of judgment is the categorical judgment in which two concepts are so linked; however, he officially recognizes other forms of judgment as well, in particular hypothetical and disjunctive judgments, in which other judgments are the representations, which are represented as linked to one another. Still, explanation (A) of the analytic/synthetic distinction in terms of the containment of the predicate concept in the subject concept applies directly only to categorical judgments. While one could fairly easily extend this explanation to hypothetical judgments (a hypothetical judgment is analytic if the antecedent "contains" the consequent) it is not so immediately obvious that it can be extended to disjunctive judgments, let alone to the myriad other forms of judgment which modern logic has taught us to recognize. Hence one is motivated to look to Kant's other explanations of the analytic/synthetic distinction, (B) and (C), for a version of the distinction, which can apply to judgments more generally. We thus have two candidate general versions of the analytic/synthetic distinction, one of which (B) looks to the content, or consequences, of a judgment, the other (C) to its grounds, or principles.

Kant clearly held, however, that these two versions of the analytic/synthetic distinction amount to the same thing. The basis for this equivalence, I suggest, lies in the Leibnizian assumption that the paradigm of a judgment is the categorical judgment, and the corresponding centrality of the concept-containment version (A) of the distinction. This version of the distinction, in other words, can serve as the link, which unites the other two, ostensibly more general versions.

Thus suppose that we are concerned with a categorical judgment of the general form S is P.[5] This is analytic, according to (A), if the concept S contains the concept P, and it is synthetic, according to (A), if the relationship between S and P is not one of containment. Consider this distinction first from the side of the consequences/content of the judgment. If the judgment is analytic, then it merely records explicitly the relation of containment, which obtains between S and P. This relation, however, is already built into the very concepts S and P themselves. The judgment that S is P, which relates these two concepts as subject and predicate, presupposes these two concepts and hence also the relation of containment between them. In grasping these concepts, as we must if we are to make the judgment, we must already have an at least implicit grasp of this relationship between them. Therefore, the analytic judgment is merely explicative – it does nothing but bring out explicitly what is already implicitly known in grasping the concepts S and P. On the other hand, if the judgment is synthetic, then the relationship between S and P is not one of containment, and is not built into the very concepts S and P themselves. Hence the judgment that S is P does not presuppose any relation between the two concepts S and P, and it is possible to grasp these concepts without even implicitly grasping the connection between them.

One can formulate the question "Is S P?" and the answer will not be forthcoming merely on the basis of one's grasp of the concepts S and P. Hence the judgment that S is P genuinely extends our knowledge and is ampliative.

Next consider the distinction drawn under (A) from the point of view of the principles, which might underlie and justify the judgment that S is P. Again, suppose first that the judgment is analytic. Then suppose one were to deny the judgment; this would amount to asserting the compatibility of the concepts S and not-P. But since the concept S contains P as a part, this would entail the compatibility of the concepts P and not-P, which is clearly a contradiction. Hence, the judgment would be justified on the basis of the principle of non-contradiction. Alternatively, the identity of the concept P with a part of the concept S grounds the judgment. On the other hand, suppose that the judgment is synthetic. Then S does not contain P, P is not identical to any part of S, and the link between them needs to be established in some other way. Kant holds that this "other way" must ultimately involve intuition, that is, the representation of a single object in which the concepts S and P are linked. Hence the principle of all synthetic judgments involves the conditions for the unity of the manifold of intuition in an experience of objects.

These considerations show why Kant could suppose that his three characterizations of the analytic/synthetic distinction are equivalent to one another. However, these arguments depend quite clearly on the treatment of categorical judgments as the paradigm of all judgments, and on the assumption of the primacy of Aristotelian logic that goes along with this. With this background in mind, we now turn to Frege.

2. Frege

In section 3 of the Foundations of Arithmetic, Frege takes up Kant's analytic/synthetic distinction. He states that the distinction concerns "not the content of the judgment but the justification for making the judgment." He adds in a footnote that "I do not mean to assign a new sense to these terms, but only to state accurately what earlier writers, Kant in particular, have meant by them." Yet Frege's remarks here seem to fly in the face of Kant's claim in the Prolegomena that the analytic/synthetic distinction concerns not the origin, but the content of the judgment.

In trying to understand what he is doing here, it is helpful to recall that, from his earliest published works, Frege showed an interest in Kant's claim that mathematical knowledge was synthetic a priori and hence required support from pure intuition. Frege opens his 1873 doctoral dissertation with the statement that "the whole of geometry rests ultimately on axioms which derive their validity from the nature of our intuitive faculty." (CP, 1.) He remarks in the first paragraph of his dissertation for the Venia Docendi of 1874: "a concept as comprehensive and abstract as the concept of quantity cannot be an intuition. There is accordingly a noteworthy difference between geometry and arithmetic in the way in which their fundamental principles are grounded. The elements of all geometrical constructions are intuitions, and geometry refers to intuition as the source of its axioms. Since the object of arithmetic does not have an intuitive character, its fundamental propositions cannot stem from intuition either." (CP, 56-7.) These comments already prefigure Frege's later argument in the Foundations for the logical character of the truths of arithmetic, based on their universal applicability, "governing the widest domain of all," so that "denying any one of them" leads to "complete confusion."[6] (FA, 21; CP, 112.)

In the Preface to his pioneering logical work, the Begriffsschrift of 1879, Frege sets out as his ultimate goal the project of determining the epistemological status of arithmetical truths – investigating whether they could be proved on the basis of logical laws alone, or needed support from some other quarter, such as Kantian pure intuition. In order to answer this question, Frege saw it as necessary to construct a new logical system in which all proofs could be carried out so as to avoid all "gaps" and to display explicitly all presuppositions and assumptions employed. In devising this logical notation, Frege tells us, he tried to express only that which is relevant to inference, which he called "conceptual content." (CN, 104.) In paragraph 3 of Begriffsschrift, Frege explains this notion through an example: "At Plataea the Greeks defeated the Persians" and “At Plataea the Persians were defeated by the Greeks” have the same conceptual content, since “the consequences which can be derived from the first judgment combined with certain others can always be derived from the second judgment combined with the same others [and vice-versa].” (CN, 112.) Here, Frege is simply following Kant’s lead in the Logic: content is a matter of what is implied by or contained in a given claim, as “the ground of many and great consequences.”

Now the question that Frege posed in the preface to the Begriffsschrift concerning the source of our knowledge of arithmetic is answered in Foundations by arguing that arithmetical truths are deducible from logical principles along with definitions of basic arithmetical concepts in logical terms. This, of course, marks Frege’s basic disagreement with Kant, and as Frege himself notes, this disagreement amounts to a disagreement over whether logic is epistemically sterile, or can extend our knowledge. For Frege, not only is logic, through its development in mathematics, ampliative, it also gives us special access to objects which are given to us independent of even the forms of our sensibility, the numbers. Frege thus sees logic as having a genuine content. Of course, Frege is able to make this sort of claim for logic in part because he has a broader conception of logic, and especially of the fundamental logical forms of judgment, one which allows him to jettison Kant’s concept-containment account (A) of the analytic/synthetic distinction as too narrow. Having rejected this account of the distinction, he is left with (B) and (C) as versions of the distinction, but no longer has the crucial link between the two that might convince him of their equivalence. Absent further argument, it is possible that there be judgments which are founded on the principle of contradiction – such that to deny them, is in effect, to contradict oneself – and which are nonetheless ampliative.

Frege chose to mark this disagreement with Kant using Kant’s own “analytic/synthetic” terminology. Given a desire to draw a distinction between analytic and synthetic judgments, then, it seems Frege must choose between the version (B) of the distinction, which focuses on the content/consequences of a judgment, and the version (C), which focuses on the principles underlying/justifying the judgment. Frege's interest in the question of the sources of arithmetical knowledge, and especially in the question of the role of intuition in mathematical knowledge, fits naturally with his choice of (C), again following Kant's lead in the Logic: "The problem becomes... that of finding the proof of the proposition, and of following it right back up to the primitive truths. If, in carrying out this process, we came only on general logical laws and on definitions, then the truth is an analytic one... If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one." (FA, 3.) Here, given that the proof of an analytic judgment depends merely on logic and definitions, there is a tolerably clear sense in which to deny such a judgment would be to contradict oneself; on the other hand, insofar as a synthetic judgment requires for its justification principles beyond those of logic (and definitions) clearly the justification must appeal to some principle other than that of contradiction. Abstractly, however, it seems equally possible to choose (B) instead, in line with Kant's Prolegomena claim that the distinction concerns the content of the judgment. Frege's disagreement with Kant could then have been expressed by claiming that logic is synthetic (that is, ampliative). Are there any deeper reasons for Frege's choice of (C)? And could they help to explain why he claims that this choice corresponds to what Kant intended?

A simple – as we will eventually see, overly simple – explanation for Frege's choice of (C) derives from his desire to retain agreement with Kant on the view that there was an interesting contrast to be drawn between analytic and synthetic judgments. Frege took Kant to have made an important and correct point in dividing judgments into the three classes of analytic, synthetic a posteriori, and synthetic a priori, and claimed to accept Kant's view that geometry ought to be classed as synthetic a priori, and that the possibility of geometrical knowledge depends on a source of knowledge which can be called our pure intuition of space. (FA, 101-102.) Frege's disagreement with Kant here was only over the location of arithmetic, not over the importance of the category of the synthetic a priori itself.

Now, if Frege had decided to adopt (B) as his way of drawing a more generally applicable analytic/synthetic distinction, he would seem to have deprived himself of any way of marking his more general agreement with Kant on the importance of the analytic/synthetic distinction. For if "analytic" means "merely explicative" and "synthetic" means "ampliative", then we would seem to lose room for a distinctive category of the synthetic a priori into which to place geometry, as opposed to arithmetic. For logic itself has turned out to generate genuine content in the form of arithmetic, and so is ampliative. Hence geometry, arithmetic and logic itself would all equally count as synthetic.

3. Russell[7]

Russell seems to draw this very conclusion in Principles, writing: "Kant never doubted for a moment that the propositions of logic are analytic, whereas he rightly perceived that those of mathematics are synthetic. It has since appeared that logic is just as synthetic as other kinds of truth; but this is a purely philosophical question, which I shall here pass by." (PM, 457.) It seems that there is again a very easy explanation of what Russell is doing here; he is simply adopting alternative (B) of the possible understandings of the analytic/synthetic distinction mentioned above, and then inferring from the ampliative nature of logic, as exhibited in the derivation of mathematics from logic, to the syntheticity of logic.

This interpretation is bolstered by the fact that Russell claims in Principles that all of mathematics is derivable from logic, including geometry. Thus, Russell had no need to make room, as Frege did, for an interesting class of the synthetic a priori into which to fit geometry as opposed to arithmetic, and so there was no obstacle to his taking the analytic/synthetic distinction, as Kant claimed, as having to do with the content of the judgment, in line with (B).

However, he drew from this the radical conclusion that no interesting distinction is marked by that between analytic and synthetic judgments at all. "Logic is just as synthetic as other kinds of truth" at least suggests that all truths are synthetic. In his Philosophy of Leibniz (1898), to which Russell, in Principles, refers in a footnote for further discussion of the analytic/synthetic distinction, he writes that supposed instances of analytic judgments are either "easily seen to be not truly analytic" or "are tautologous, and so not properly propositions at all." (PL, 16-17.) By "tautologous" here Russell means propositions of the form "A is A," bare identities – a usage which goes back to Kant's Logic. Kant writes that "the identity of the concepts in analytic judgments can be either explicit ... or non-explicit .... In the first case the analytic propositions are tautological." He adds that "Tautological propositions are empty virtualiter, or empty of consequences, for they are without value or use. The tautological proposition Man is man, is of this sort, for example." (LL, 607.) The contrast here is with non-tautological analytic propositions, which are devoid of content in the sense of not really extending our knowledge but which are "not empty of consequences or fruitless for they make clear the predicate that lay undeveloped ... in the concept of the subject through development." Kant allows that such propositions "increase cognition formaliter" but not "materialiter" so that we could say that non-tautological analytic propositions are "empty materialiter" but not "empty virtualiter." Russell's claim is that Kant's supposed contrast between tautological and non-tautological analytic propositions is void – even the non-tautological propositions of logic are synthetic.

Thus we seem to have a simple explanation of the divergences between Russell and Frege in their responses to Kant. However, while simplicity is a virtue, we must also be wary of over-simplification, bearing in mind Austin's remark: "You will have heard it said, I suspect, that over-simplification is the occupational disease of philosophers, and in a way one might agree with that. But for a sneaking suspicion that it is their occupation." (Austin 1979, 252.) And indeed, on closer examination, we will find that the above sketch does present too simplistic a view of the relationship of Kant, Frege and Russell's views on the analytic/synthetic distinction.

To begin with, it is to be noted that the appearance of a disagreement concerning the status of geometry, between Kant and Frege on the one hand, and Russell on the other, is exaggerated. For Russell's claim that "all Mathematics is Symbolic Logic," where "Mathematics includes not only Arithmetic and Analysis, but also Geometry, Euclidean and non-Euclidean, rational Dynamics, and an infinite number of other studies still unborn or in their infancy" (PM, 4-5) depends on his explanation of "pure mathematics" as "the class of all propositions of the form 'p implies q,' where p and q are propositions containing one or more variables, and neither p nor q contains any constants except logical constants." (PM, 3.) This "if-then-ism" is adopted by Russell in part to get around the fact that "the actual propositions of Euclid... do not follow from the principles of logic alone," a fact which he says "led Kant to his innovations in the theory of knowledge." (PM, 5.) Russell's definition of pure mathematics, however, rules the "actual propositions of Euclid" outside the boundaries of pure mathematics, so that "in pure mathematics, the Euclidean and non-Euclidean geometries are equally true: in each nothing is affirmed except implications." (PM, 5.) On this score, Frege would quite happily concede that Euclidean and non-Euclidean geometries so understood are equally logically true. But he would further claim that Euclidean geometry, understood as consisting in (roughly) the "actual propositions of Euclid" is also known to be true, but not on the basis of logic.

This is only the beginning of the story. To get a better understanding of the real roots of Russell's claim that logic is synthetic, we need to turn to a more careful look at the passages from the Philosophy of Leibniz in which he discusses the analytic/synthetic distinction. What will emerge is that Russell is concerned with the way in which analytic judgments are said to rest on the analysis of concepts, whereas synthetic judgments depend on a synthesis. Russell's argument will be that all propositions except bare tautologies depend on a synthesis in the formation of concepts, and so are synthetic. I will argue that an examination of this argument shows that Russell's attitude towards the role of logic in the analysis of concepts is quite different from Frege's. As a result, I will conclude, it is misleading to think of Frege and Russell as engaged in a common project, "logicism," in their two early works; for their conceptions of logic itself are importantly different.

Russell's discussion of the analytic/synthetic distinction in the Philosophy of Leibniz occurs in section 11. It takes place against the background presupposition that all judgments are ultimately of subject-predicate form, a view that Russell attributes to Leibniz and discusses critically in section 10. While Russell clearly rejects such a view in Principles, it is noteworthy that he there provides no discussion of the analytic/synthetic distinction suited to this rejection, merely referring the reader to his earlier treatment. In this treatment he begins by criticizing what he takes to be Leibniz's understanding of the distinction, arguing that Kant correctly took arithmetic and geometry to be synthetic; but he ends by concluding that "the doctrine of analytic propositions seems wholly mistaken" and that even such seemingly analytic propositions as "the equilateral rectangle is a rectangle" are synthetic rather than analytic. (PL, 22-23.) He employs a number of arguments to this conclusion, but all have, I think, the same basic structure: the thought is that an analytic judgment S is P must be based on an analysis of the subject-concept S which reveals that the predicate-concept P is contained in it; but this analysis presupposes a prior synthesis in which the concept S is assembled out of its various parts, including P. Consequently, Russell concludes, the judgment S is P is not purely analytic, but really at bottom involves or depends on synthetic judgments. This basic argument has a vaguely Kantian ring; one is reminded of Kant's claim in the B-deduction that "the dissolution (analysis) that seems to be its [synthesis's] opposite, in fact always presupposes it; for where the understanding has not previously combined anything, neither can it dissolve anything, for only through it can something have been given to the power of representation as combined." (CPR, B130)

Russell's first version of this argument begins by claiming that in any analytic judgment S is P, the subject-concept S must be "a complex idea, i.e. a collection of attributes, while the predicate is some part of this collection." He then goes on to argue that "the collection, however, – and this is the weak point of the doctrine of analytic judgments – must not be any haphazard collection, but a collection of compatible or jointly predicable predicates... Now this compatibility, since it is presupposed by the analytic judgment, cannot itself be analytic." (PL, 18.) Here Russell is assuming that the subject in an analytic judgment must at least be a possible predicate of some individuals, and thus the analytic judgment presupposes the synthetic judgment that the predicates combined in the subject-concept are compatible.

Russell then turns to a version of the argument couched in terms of definition. He claims that "definition ... is only possible in respect of complex ideas" and "consists, broadly speaking, in the analysis of complex ideas into their simple constituents." He then argues that this account of definition "is inconsistent with the doctrine that the 'primary principles' are identical or analytic; and that the former is correct, while the latter is erroneous." His argument turns on the Leibnizian claim that "the objects of definitions must be shown to be possible." Thus "there is always involved, in definition, the synthetic proposition that the simple constituents are compatible." (PL, 18-19.) Simple ideas, in and of themselves, and without presupposing synthetic relations of compatibility and incompatibility, are never contradictory, Russell claims, and so "we may argue generally, from the mere statement of the Law of Contradiction, that no proposition can follow from it alone, except the proposition that there is truth, or that some proposition is true." (PL, 20.)

Russell's claim in these arguments that the relation of compatibility between concepts is a "synthetic relation" is interestingly related to a claim made in the Principles, that the relation of implication between propositions is also synthetic: implication is, after all, essentially the opposite of compatibility (A implies B just in case A is incompatible with not-B). Russell writes that "implication is a synthetic relation," explaining that while "if A be an aggregate of propositions, A implies any proposition which is a part of A, it by no means follows that any proposition which A implies is part of A." (Principles, 349.) James Levine, in an interesting discussion of Russell and Frege's conceptions of analysis, cites this passage as providing the explanation of Russell's claim that logic is synthetic. (Levine 2002, 215-216.) This passage, however, leaves open the possibility of at least some analytic judgments that are other than bare identities, such as "the equilateral rectangle is a rectangle," and therefore also leaves open the possibility of some logical principles being analytic, namely at least such principles as "A and B implies A." This possibility is ruled out by the final argument of section 11 of The Philosophy of Leibniz.

This final and most general form of the argument occurs in the concluding paragraph of section 11. Russell there concludes: "even those propositions which, at the beginning of the enquiry, we took as the type of analytic propositions, such as 'the equilateral rectangle is a rectangle,' are not wholly analytic." Not only do they presuppose "synthetic propositions asserting that the constituents of the subject are compatible," in and of themselves "they are judgments of whole and part." Thus, "the constituents, in the subject, have a certain kind of unity – the kind always involved in numeration, or in assertions of a whole – which is taken away by analysis. Thus even here, in so far as the subject is one, the judgment does not follow from the Law of Contradiction alone." (PL, 22-23.)

Two points are notable about this Russellian argument, which, as I have pointed out, is the only reference directly given by Russell in Principles for the justification of the claim that "logic, like other kinds of truth, is synthetic." First, it does not turn on a conception of the analytic/synthetic distinction as the ampliative/explicative distinction. Rather it turns on considerations of the processes of analysis and synthesis that would justify such judgments and involves the argument that in all but bare tautologous self-identities synthesis is presupposed. Second, it remains wedded to a building-block picture of concepts in which simple concepts are combined with one another by Boolean logical operations and then taken back apart by analysis. Here we encounter the real deep and abiding difference between the early Russell and the early Frege's conceptions of the power of the new logic.

4. Concepts, judgments, and analysis in Frege and Russell[8]

Earlier, I suggested an explanation of Frege's choice of version (C) of the analytic/synthetic distinction as required to secure logical space for recording his agreement with Kant on the synthetic a priori status of geometry. However, a deeper reason for Frege's choice of (C) reveals itself if, following out our discussion of Russell, we think of the classification of judgments into "analytic" and "synthetic" as dependent on distinct forms of justification of the judgments so classified, labeled "analysis" and "synthesis" respectively. If we focus on analytic judgments in particular, it is important to realize that Frege's new logic made possible a new form of conceptual analysis.

Frege does not himself speak directly of the "analysis" (Zerlegung) of concepts very often;[9] he prefers to speak of the "formation" (Bildung) of concepts. However, this Bildung of concepts should not be thought of as a psychological process through which concepts are generated. Rather, it can be understood as the perspicuous representation of concepts, in the sense in which a Bild is a picture or model.[10] Frege thought of his new logic as providing a tool capable of revealing the structure of concepts, structure that could not be made explicit using the simple part-whole analysis of concepts suggested by Kant's Logic and equally presupposed in Boolean accounts. Frege's very choice of a name for his notation, "Begriffsschrift," or "concept-writing," indicates the importance of this idea in his early thought about logic. In a long polemical essay comparing his Begriffsschrift to Boole's logical notation, written in 1880 or 1881, Frege says: "Right from the start I had in mind the expression of a content." But, he adds, "…the content is to be rendered more exactly than is done by verbal language," where "there is only an imperfect correspondence between the way the words are concatenated and the structure of the concepts. The words 'lifeboat' and 'deathbed' are similarly constructed though the logical relations of the constituents are different." He goes on to remark that arithmetic "forms concepts … of such richness and fineness in their internal structure that in perhaps no other science are they to be found combined with the same logical perfection." (PW, 12-13.) Similarly, in the Introduction to Foundations, he writes: "the concept of number… has a finer structure than most of the concepts of the other sciences." (FA, iv.) The successful application of the Begriffsschrift to arithmetic will provide an expression for arithmetical concepts in which the fine structure of these concepts will be made perspicuous.

At the same time, the expression of arithmetical concepts in Begriffsschrift will make clear their richness, their content as the "ground of many and great consequences." This is why Frege, in Foundations, claims that his definitions possess the virtue of fruitfulness.[11] (FA, 81, 100-101.) Frege's definitions of fundamental arithmetical concepts are based on careful reflection on the logical relations in which those concepts are involved, and so of their content. This kind of complex analysis of concepts is exhibited in such Begriffsschrift definitions as those of the ancestral of a relation and of a many-one relation. These definitions, in turn, are made possible by the crucial innovation of Frege's logic in adopting the device of quantifiers and bound variables to represent logical structures involving relations and embedded quantification. The process of extracting from concepts like that of the ancestral of a relation and that of a many-one relation, the kinds of consequences exhibited in theorems like the last proposition of the Begriffsschrift, which states that the ancestral of a many-one relation is connected, can be seen as a process of developing the implicit results of a logical analysis of the structure of the concepts involved.

Thus Frege's Begriffsschrift made possible a new form of conceptual analysis. In the Preface to the Begriffsschrift, he connects this to his innovative logical analysis of the contents of judgment in terms of function and argument: "the replacement of the concepts of subject and predicate by argument and function will prove itself in the long run. It is easy to see how regarding a content as the function of an argument leads to the formation of concepts." (CN, 107.) In the Begriffsschrift Frege explains the notion of a function on a linguistic level: "if, in an expression... a simple or complex symbol occurs in one or more places and we imagine it as replaceable by another (but the same one each time) at all or some of those places, then we call the part of the expression that shows itself invariant a function and the replaceable part its argument." (CN, 127.)

In writings preceding the publication of Foundations, and in Foundations itself, Frege extends this replacement and omission model of the generation of function-expressions from complete expressions, to the generation of concepts from judgeable contents. In a letter written in

1882 he writes: “I do not believe that concept formation can precede judgment because this would presuppose the independent existence of concepts, but I think of a concept as having arisen by decomposition from a judgeable content.” (PMC, 101.) Similarly, in his 1880/81 polemical essay, he contrasts his view of concept-formation with Boole’s:

> in Boole, the logically primitive activity is the formation of concepts by abstraction, and judgment and inference enter in through an activity of immediate or indirect comparison of concepts via their extensions. ... As opposed to this, I start out from judgments and their contents, and not from concepts. ... I only allow the formation of concepts to proceed from judgments. If, that is, you imagine the 2 in the judgeable content
>
> $2^4 = 16$
>
> to be replaced by something else, by (-2) or by 3 say, which may be indicated by putting an x in place of the 2:
>
> $x^4 = 16$,
>
> the judgeable content is thus split up into a constant and a variable part. The former, regarded in its own right but holding a place open for the latter, gives the concept ‘4^{th} root of 16’. (PW, 15-16.)

In Foundations itself, Frege tells us that “if, from a judgeable content which deals with an object a and an object b we subtract a and b, we obtain as remainder a relation-concept, which is, accordingly, incomplete at two points.” (FA, 82) As this quotation shows, Frege drew a consequence from his account of the formation of concepts from judgeable contents: concepts are “incomplete” or “unsaturated.” As he put it in the letter from 1882 cited above: “A concept is unsaturated in that it requires something to fall under it; hence it cannot exist on its own.” (PMC, 101.) Frege hence draws a very sharp distinction between concepts and objects, the latter being characterized as “self-subsistent.” In the Foundations, the third of Frege’s guiding methodological principles is “never to lose sight of the distinction between concept and object.” (FA, x.) The key to this distinction is the distinction between objects as “complete,” “saturated,” or “self-subsistent” entities which can serve as the arguments of functions, and concepts as functions, the “incomplete,” “unsaturated” entities which require completion by arguments.

In the conclusion of Foundations, as well as in his polemic against Boole, Frege emphasized that his method of generating concepts out of judgeable contents was more fruitful and logically valuable than the method of constructing concepts out of already given concepts by means of Boolean operations. (FA, 99-101; PW, 32-34.) In Foundations, Frege criticizes Kant for adopting such a conception: “he seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful.” (FA, 100.) Earlier, Frege had emphasized that “definitions show their worth by proving fruitful” (FA, ix, 81) and that “the concept of number ... has a finer structure than most of the concepts of the other sciences, even although it is one of the simplest in arithmetic.” (FA, iv.) In his conclusion, he asserts that in “the really fruitful definitions in mathematics... What we find ... is not a simple list of characteristics; every element in the definition is intimately, I might almost say organically, connected with the others.” He offers a “geometrical illustration:”

> If we represent the concepts... by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics. With a definition like this, therefore, what we do ... is to use the lines already given in a new way for the purpose of demarcating an area.”

This passage is drawn almost verbatim from his polemic against Boole, where he makes the same point about Boolean combinations of concepts in general (although he speaks in terms of “concept formation” rather than “definition”). In contrast, he says in Foundations, “the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic."[12] (FA, 100-101.)

An example might help to make clear what Frege has in mind.[13] Consider the following definition of prime number:

n is prime : n □1 & □m(□k(km = n) 6 (m=1 ω m=n)).

Here we have defined "prime" in terms of multiplication, one, and logical vocabulary, but there is no sense in which this definition is simply given by a list (or even a Boolean combination) of characteristics or relations (product of, identical with...) which together make up the concept of a prime number. Rather, we have given a complex pattern, which can be applied to any number n to produce a statement asserting that n is prime. Note two features of this procedure: first, the pattern we have produced makes explicit use of the apparatus of bound variables and multiple quantification which is the key innovation of Frege's logic; and second the pattern we have produced amounts precisely to a specification of the concept of prime number as a function in Frege's sense. With such a definition, then, we have a form of analysis in which we begin, really, with complex judgments like:

2 □1 & □m(□k(km = 2) 6 (m=1 ω m=2))

3 □1 & □m(□k(km = 3) 6 (m=1 ω m=3))

5 □1 & □m(□k(km = 5) 6 (m=1 ω m=5))

and recognize in them a common pattern, the concept of prime number

n □1 & □m(□k(km = n) 6 (m=1 ω m=n)).

And with this form of analysis, we cannot anticipate in advance, Frege claims, what we will be able to prove about the concept of prime number. "We are not simply taking out of the box again what we have just put into it."

Furthermore, what is central in the definition of "prime number" exhibited above is that this definition lays bare the logical content of the concept, showing what would follow from an application of the concept to a number n. In particular, it makes explicit that from the claim that n is prime, together with the claim that km = n, we can infer that either m=1 or m=n. The value of analysis resides in bringing to our attention such patterns of inference, patterns of which judgments follow from which other judgments. Thus we see why Frege claims to start with judgments and their contents, and derive concepts from them, and why he claims that this mode of concept-formation is "fruitful."

Thus, Frege's notion of "analyticity" is tied to a new conception of the analysis of concepts. Now, if we return to Russell's arguments in the Philosophy of Leibniz for the claim that all judgments but trivial identities are synthetic, we can see immediately that it relies on a conception of analysis much more like that which Frege criticizes in Kant and Boole. The key idea in all of Russell's arguments, as we saw, was that analysis always presupposes a prior synthesis, that we cannot analyze what we have not already combined. On this view it is clear that we will always in analysis be "simply taking out of the box again what we have just put into it." Russell takes it (in this early period) that every unity is put together out of independent constituents. The unity of the proposition then becomes a problem for Russell. Frege, on the other hand, takes for granted the unity of judgments and their contents. He envisages a mode of analysis of judgments into concepts and objects that is not merely a matter of taking apart something, which we have previously assembled. Rather, it is a matter of recognizing inferential relations among judgments, and providing an analysis that lays bare and systematizes those relations.

5. Preliminary conclusions: logic in Frege and Russell's logicisms

What then can we conclude from all this? I think that we have found enough differences between Frege and Russell's conceptions of the "logicist project" to suggest that it is misleading at best to think of them as engaged in a common project. Most importantly, (early) Frege and (early) Russell had sufficiently different conceptions of "logic," so that the slogan "mathematics is part of logic" did not really have the same force for each of them. In this concluding section, I offer some preliminary remarks on Frege's and Russell's different conceptions of the importance of the new "symbolic logic" to logic, and so to their respective "logicisms."[14]

To begin with, it is significant that Frege invented his Begriffsschrift, whereas Russell learned what he called "Symbolic Logic" from his study of others' works, first from Whitehead, then from Peano, and finally from Frege. Frege, in a 1906 fragment titled "What may I regard as the Result of my Work?" wrote: "It is almost all tied up with the concept-script. A concept construed as a function." (PW, 184) Frege's logical discoveries were deeply intertwined with the creation of the Begriffsschrift. He viewed this symbolism as making perspicuous key logical distinctions which ordinary language covered up. His attitude towards the relative merits of ordinary language and his conceptual notation comes out clearly in the preface to the Begriffsschrift:

> If it is a task of philosophy to break the power of the word over the human mind, uncovering illusions which through the use of language often almost unavoidably arise concerning the relations of concepts, freeing thought from that which only the nature of the linguistic means of expression attaches to it, then my Begriffsschrift, further developed for these purposes, can become a useful tool for the philosopher. (CN, 106)

Among the illusions fostered by language, the most important for Frege was the refusal to recognize a sharp distinction between concept and object. In his polemic against Boolean logic, discussed above, he emphasized that the Begriffsschrift always respects the distinction between unsaturated concept and saturated object:

> ...in the Begriffsschrift their designations [the designations of concepts] never occur on their own, but always in combinations which express judgeable contents. ... A sign for a property never appears without a thing to which it might belong at least indicated ... (PW, 17.)

Even in a quantificational context like "□xFx" the concept-expression "F( )" is completed by the letter "x", which indicates an argument for the concept and so shows its unsaturatedness.

In his unpublished polemic against Boole, Frege described his Begriffsschrift as a "fresh approach to the Leibnizian idea of a lingua characterica," a "universal language." Frege distinguished this idea from the closely associated one of a "calculus ratiocinator" or calculus of reason.[15] Using these terms, he argued that his Begriffsschrift had fundamentally different aims from those of Boole's logic (PW, 12-13):

> ...it is necessary that we should always bear in mind the purpose that governed Boole in his symbolic logic and the one that governed me in my Begriffsschrift. If I understand him aright, Boole wanted to construct a technique for resolving logical problems systematically, similar to the technique of elimination and working out the unknown that algebra teaches. ... In all this there is no concern about content whatsoever. ... In contrast we may now set out the aim of my concept-script. Right from the start I had in mind the expression of a content. What I am striving for is a lingua characterica in the first instance for mathematics, not a calculus restricted to pure logic. But the content is to be rendered more exactly than is done by verbal language. For that leaves a great deal to guesswork, even if only of the most elementary kind. There is only an imperfect correspondence between the way words are concatenated and the structure of the concepts. ... A lingua characterica ought, as Leibniz says, peindre non pas les paroles, mais les pensées [to depict not words, but thoughts].

For Frege, a lingua characterica such as his Begriffsschrift is an instrument for overcoming language-induced confusion, and hence for "breaking the power of the word over the human mind." Boolean logic is not such an instrument, but rather a mere calculus ratiocinator.

In this connection, it is interesting that Russell was first introduced to "symbolic logic" through his reading of Whitehead's presentation of Boolean logic in his A Treatise on Universal Algebra of 1898. Whitehead opens this work with a general discussion of signs. He distinguishes between the "expressive signs" of "ordinary language" and the "substitutive signs" of algebra:

> In the use of expressive signs the attention is not fixed on the sign itself but on what it expresses; that is to say, it is fixed on the meaning conveyed by the sign. Ordinary language consists of groups of expressive signs...
>
> A substitutive sign is such that in thought it takes the place of that for which it is substituted. A counter in a game may be such a sign: at the end of the game the counters lost or won may be interpreted in the form of money, but till then it may be convenient for attention to be concentrated on the counters and not on their signification. The signs of a Mathematical Calculus are substitutive signs.
>
> The difference between words and substitutive signs has been stated thus, 'a word is an instrument for thinking about the meaning which it expresses; a substitute sign is a means of not thinking about the meaning which it symbolizes.' (UA, 3-4)

Whitehead goes on to introduce the notion of a Calculus:

> In order that reasoning may be conducted by means of substitutive signs, it is necessary that rules be given for the manipulation of the signs. ...
>
> The art of the manipulation of substitutive signs according to fixed rules, and of the deduction therefrom of true propositions, is a Calculus. ...
>
> When once the rules for the manipulation of the signs of a calculus are known, the art of their practical manipulation can be studied apart from any attention to the meaning to be assigned to the signs. ... (UA, 4-5)

He reiterates the thought that such a calculus allows us to avoid thinking:

> Not only can the reasoning be transferred from the originals to the substitutive signs, but the imaginative thought can in large measure be avoided. ... A calculus avoids the necessity for inference and replaces it by an external demonstration... (UA, 10)

Whitehead's first example of an algebra is Boolean algebra. He develops this in chapters I-III of Book II, turning only in Chapter IV to its "Application to Logic." He begins his discussion with the following comment:

> It remains to notice the application of this algebra to Formal Logic, conceived as the Art of Deductive Reasoning. It seems obvious that a calculus – beyond its suggestiveness – can add nothing to the theory of Reasoning. For the use of a calculus is after all nothing but a way of avoiding reasoning by the help of the manipulation of symbols. (UA, 99.)

Thus, for Whitehead, Boole's logical symbolism cannot be essential to logic, the science of reasoning; it is at best a "suggestive" heuristic aid in reasoning.

In his Philosophy of Leibniz of 1900, Russell discussed symbolic logic in connection with Leibniz's project for a "universal characteristic:"

> This was an idea which he cherished throughout his life ... He seems to have thought that the symbolic method, in which formal rules obviate the very necessity of thinking, could produce everywhere the same fruitful results as it has produced in the sciences of number and quantity. ... What he desired was evidently akin to the modern science of Symbolic Logic, which is definitely a branch of Mathematics, and was developed by Boole under the impression that he was dealing with the "Laws of Thought." As a mathematical idea – as a Universal Algebra, embracing Formal Logic, ordinary Algebra, and Geometry as special cases – Leibniz's conception has shown itself in the highest degree useful. (PL, 169-170, my emphasis).

Here Russell assimilates Leibniz's project for a universal characteristic to the mechanically operable calculi of Whitehead's Universal Algebra, whose purpose is to make reasoning unnecessary. This contrasts with the view of Frege, who, as we saw, viewed the universal characteristic as including not only a "calculus ratiocinator" but also a "lingua characterica" for the perspicuous representation of thoughts. For Frege, the Begriffsschrift is not a means for avoiding thought, but a means for thinking clearly – for the first time.

The attitude towards "symbolic logic" which Russell learned from Whitehead and which was manifested in the Philosophy of Leibniz persists in the Principles of Mathematics. At the end of the first chapter of that work, Russell states that "the present work has to fulfill two objects, first, to show that all mathematics follows from symbolic logic, and secondly to discover, as far as possible, what are the principles of symbolic logic itself." (PM, 9.) Yet he begins his second chapter on "Symbolic Logic" by announcing: "the word symbolic designates the subject by an accidental characteristic, for the employment of mathematical symbols, here as elsewhere, is merely a theoretically irrelevant convenience." (PM, 10.) In fact, in this chapter of Principles, Russell manages to present the "principles of symbolic logic" virtually without using symbols!

The logic that he presents there, however, is not Boole's but Peano's. In "My Mental Development," written in 1944, Russell described his discovery of Peano's logic during his visit to the International Congress of Philosophy in Paris in 1900 as "the most important event" in "the most important year in my intellectual life." (Russell 1958, 12.) Russell's first major work in logic followed this discovery: a presentation of the Peirce-Schröder logic of relations in Peano's notation, published in Peano's house journal, the Revue des Mathématiques (earlier the Rivista di Matematica) in 1901.

Peano, like Frege, often introduced his notation in relation to Leibniz's project for a "universal characteristic:" "After two centuries, this 'dream' of the inventor of the infinitesimal calculus has become a reality. ... We have therefore the solution to the problem proposed by Leibniz." (OS, 196.) Again like Frege, he portrays his notation as not merely a calculus but also a language: it is "capable of representing all the ideas of logic, so that by introducing symbols to represent the ideas of the other sciences, we may express every theory symbolically." It is "not just a conventional abbreviated writing, ... since our symbols do not represent words but ideas." (SW, 190-191.)

Shortly after his introduction to Peano's logical notation, Russell celebrated it as a realization of Leibniz's dream. In a survey of Italian logical work published in Mind in 1901, he wrote that

> To Peano … is due the revival, or at least the realization, of Leibniz=s great idea, that, if symbolic logic does really contain the essence of deductive reasoning, then all correct deduction must be capable of exhibition as a calculation by rules. (CPBR3, 353)

Here, as in the Philosophy of Leibniz, Russell's emphasis is on the role that a logical language can play in reducing deduction to calculation. Similarly, in a popularizing article, "Recent Work on the Principles of Mathematics," Russell again credits Peano with realizing Leibniz's grand project:[16]

> Two hundred years ago, Leibniz foresaw the science which Peano has perfected, and endeavored to create it. He was prevented from succeeding by respect for the authority of Aristotle, whom he could not believe guilty of definite, formal fallacies; but the subject which he desired to create now exists, in spite of the patronizing contempt with which his schemes have been treated by all superior persons. From this AUniversal Characteristic,@ as he called it, he hoped for a solution of all problems, and an end of all disputes. … over an enormous field of what was formerly controversial, Leibniz=s dream has become sober fact. (CPBR3, 369)

But Russell's explanation of the virtues of Peano's notation again emphasizes its aspect as a calculus, an algorithm:[17]

> People have discovered how to make reasoning symbolic, as it is in Algebra, so that deductions are effected by mathematical rules. …It is not easy for the lay

> mind to realize the importance of symbolism in discussing the foundations of mathematics, and the explanation may perhaps seem strangely paradoxical. The fact is that symbolism is useful because it makes things difficult. ... What we wish to know is, what can be deduced from what. Now, in the beginnings, everything is self-evident, and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy to correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical. ... The great master of the art of formal reasoning, among the men of our day, is an Italian, Professor Peano, of the University of Turin. (CPBR3, 367-8)

A few years earlier, Frege had published an article comparing his Begriffsschrift with Peano's notation, and a letter from Frege to Peano concerning the relation between their symbolisms was published in Peano's journal, Rivista di Matematica, along with Peano's reply. In his published article, Frege takes up the "respective aims" of their notations, as he had in comparing his notation to Boole's. (CP, 234.) He concludes that Peano's "enterprise more closely resembles my conceptual notation than it does Boole's logic" since it is intended both as a "lingua characterica" and a "calculus ratiocinator." (CP, 242.) Nonetheless, Frege criticizes many of the details of Peano's notation, claiming that "the striving for logical perfection is less marked here than in my Begriffsschrift." (CP, 238.) One of the advantages Frege cites for the Begriffsschrift is again its marking of the distinction between concept, or more generally function, and object: "I distinguish function-letters from object-letters, using the former to indicate only functions and the latter to indicate only objects, in conformity with my sharp differentiation between functions and objects, with which Mr. Peano is unacquainted." (CP, 248.) In the letter published in the Rivista di Matematica, Frege provides an example of a proposition from Peano's Formulario in which "the function letter 'f' occurs ... without an argument place." He writes: "this is to misunderstand the essence of a function, which consists in its need for completion. One particular consequence of this is that every function sign must always carry with it one or more places which are to be taken by argument signs; and these argument places – not the argument signs themselves – are a necessary component part of the function sign." (PMC, 115-116, my emphasis.)

Russell's "The Logic of Relations" of 1901 reproduces what would be for Frege the same error, in its very first proposition:

1 R , Rel. ε :xRy. = . x has the relation R with y. (CPBR3, 315.)

Here "R" occurs both with and without argument places – for Frege an instance of equivocation between concept and object. In Principles, however, Russell rejects this Fregean distinction. He relies on the grammar of ordinary language to guide his reflections, rather than the structure of "symbolic logic," writing that:

> The study of grammar, in my opinion, is capable of throwing far more light on philosophical questions than is commonly supposed by philosophers. Although a grammatical distinction cannot be uncritically assumed to correspond to a genuine philosophical difference, yet the one is prima facie evidence of the other, and may often be most usefully employed as a source of discovery. ... On the whole, grammar seems to me to bring us much nearer the correct logic than the current opinions of philosophers; and in what follows, grammar, though not our master, will yet be taken as our guide. (PM, 42.)

Following this guide, Russell adopts an ontology in the Principles which is based on the fundamental principle that "every constituent of every proposition must, on pain of self-contradiction, be capable of being made a logical subject." (PM, 48.) According to this ontology there is a most general category of entities, the category of terms. Propositions (the equivalent of the early Frege's judgeable contents) are complex terms composed of simple terms as their parts. Given any proposition, some of the terms that make it up are distinguished as "the terms of the proposition," its "logical subjects" – these are the terms, which the proposition can be said to be "about." For example, in the proposition Socrates is human the two terms Socrates and human(ity) both occur as constituents, but only Socrates occurs as a term of the proposition, its logical subject. The terms that occur as terms of a proposition (its logical subjects) are distinguished by the fact that any term can sensibly replace them in the proposition – thus we can form not only the proposition Plato is human but also the proposition humanity is human from Socrates is human. In contrast, according to Russell, "with the sense that is has in this proposition," Socrates could not replace human. Thus the only term of Socrates is human is Socrates; this proposition is about Socrates, but not human(ity). Every term can be the subject of propositions about it, propositions in which it occurs as a term of the proposition; for example, humanity occurs as a term of the proposition humanity is a concept. Some terms, such as Socrates, can only occur as terms of the propositions in which they occur; Russell calls these "things." Other terms, such as humanity, have a "curious twofold use," occurring in some propositions without occurring as terms of those propositions; these are called "concepts." (Principles, 44-45.)

Now here Russell does have a distinction between "things" and "concepts." But by insisting that both "things" and "concepts" can be captured under the more general heading "terms," and that any term can be made the logical subject of a proposition, so that any term, even a concept, can sensibly replace a thing in a proposition, Russell shows clearly that his distinction is not the same as that which Frege sought to draw between concept and object. For Frege, what can be intelligibly said of self-subsistent objects cannot even be intelligibly said of incomplete concepts – for example, we cannot predicate identity of concepts as we can of objects, but only an analog of the relation of identity between objects.

Russell was well aware of the difference between his account of concepts and things, and

Frege's distinction between concepts and objects, devoting several pages of his Appendix to the Principles on "The Logical and Arithmetical Doctrines of Frege" to a discussion of this issue, and concluding that "Frege's theory... will not, I think, bear investigation," and that "the doctrine of concepts which cannot be made subjects seems untenable." (Principles, 510.) I do not wish to enter into this controversy here, but only want to suggest that this provides us with a deep connection to Frege and Russell's differing views on the analytic/synthetic distinction in their early logicist works. Russell at points comes close in Principles to a conception like that of Frege's view of concepts as unsaturated functions, particularly in his discussions of assertions about a term and of propositional functions. But he remains deeply suspicious of such a conception. Consequently he never, in Principles, clearly sees what Frege saw, that the replacement of the grammatical analysis of propositions into subject and predicate with a logical analysis into function and argument "leads to the formation of concepts" (CN, 107) and thus makes possible a new and distinctive mode of conceptual analysis, one in which we do not "simply take out of the box what we have just put into it."

Thus, although Russell was clearly aware of the power of logical developments such as Frege's, initially through his encounter with Peano, and then through his own study of Frege's works, he was nonetheless prevented from appreciating the sense in which Frege could claim to have achieved a new and more fruitful conception of the formation of concepts. What prevented Russell from doing this was in part his conception of "symbolic logic" as only, insofar as it is logic, "accidentally" symbolic. "Logic" in Principles is essentially independent of any particular system of notation. Frege's "logic" on the other hand, "is almost all tied up with the concept-script." While both Frege and Russell may have claimed that mathematics is a part of logic, then, they did not intend the same thing by this thesis. Their "logicisms" were deeply, not merely superficially different.

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NOTES

[1] Versions of this paper were presented to the Philosophy of Mathematics Workshop of the University of Notre Dame (November 8, 2001), and the Philosophy Departments of the University of Chicago (November 14, 2002), and the University of California at Los Angeles (February 28, 2003). Thanks are due to helpful discussion from members of all three audiences. Special thanks are due to Joan Weiner for comments on an earlier draft of this paper.

After substantially completing work on this paper, I came across (Levine 2002), which arrives at similar conclusions to mine through a quite different route. I briefly discuss one aspect of Levine's paper in section 3 below.

[2] I emphasize "early" Frege and Russell in my title, because the development of Frege's thought after Foundations, and Russell's after Principles, involved changes in both of their conceptions of "logic," which in some ways brought them closer together, while in other ways drove them further apart. That, however, is a story for another paper.

[3] One might object, as did an anonymous referee for another journal, that there are still sufficient similarities between Frege's and Russell's views on logic to justify classifying their logicisms together: in particular, both hold that logic has no need of intuition, and both hold that logic consists of maximally general substantive truths. The first point is surely right, and underlies both Frege and Russell's responses to Kant insofar as both reject the thought that mathematical proofs rely on intuition. But it is insufficient to establish the claim that it is appropriate to treat Frege and Russell as having a common conception of logic, given the differences I detail in this paper. On the second point, however, I do not think that the conception of logical truths as maximally general substantive truths is held by the early Frege, who is (half of) the focus of this paper. John MacFarlane has argued, against Thomas Ricketts and Warren Goldfarb, that the

sense in which logic is "general" for Frege is normative rather than descriptive: "logic is general in the sense that it provides constitutive norms for thought as such, regardless of its subject matter." (MacFarlane 2002, 35; see Goldfarb 1979, 2001, and Ricketts 1985, 1986a, 1986b, 1996, for the views that MacFarlane is opposing.) MacFarlane makes this claim about Frege's view of logic throughout his career; I believe that the case can be made even more convincingly for the Frege of the Begriffsschrift and the Grundlagen. For example, in his first attempt to write a "Logic," in the early 1880s, Frege characterizes the goal of logic as establishing the "laws of valid inference" and argues that "the subject-matter of logic is therefore such as cannot be perceived by the senses…" (PW, 3) The account of logical truths as maximally general substantive truths is not on view in this fragment. This conception of the generality of logic does come to play a role in Frege's thought after the introduction of the sense-meaning distinction, but remains in tension with the earlier, normative conception, as MacFarlane argues.

[4] I draw my inspiration in this section in part from related discussions in (Tappenden 1995).

[5] The following argument works best in the case of universal affirmative judgments. I ignore complications that would be occasioned by considering other forms of categorical judgment here.

[6] In this paragraph I am indebted to correspondence with Joan Weiner.

[7] The best treatment of Russell's early philosophy remains (Hylton 1990). However, Hylton does not give as complete a treatment of Russell's arguments in the Philosophy of Leibniz as I do here.

[8] Once again, I am indebted to (Tappenden 1995), as well as to (Dummett 1991), here. See also (Kremer, forthcoming).

[9] There are, however, exceptions, for example (CN, 88).

[10] I owe this point to Joan Weiner, although she would not draw from it the same conclusion that I do.

[11] Note here Frege's use of the words "richness" and "fruitfulness," used by Kant in describing the content of a cognition.

[12] In this passage Frege appeals to both versions (B) and (C) of the analytic/synthetic distinction, recognizing the reasons for attributing (B) to Kant while insisting on taking (C) to be the correct way of drawing the distinction.

[13] This example is also used by Peter Sullivan in "Frege's Logic." (Sullivan 2004, 696)

[14]On the restriction to "early" Frege and Russell, see note 2.

[15]Frege's appeal to these Leibnizean ideas was noted by Jean van Heijenhoort in his seminal paper "Logic as calculus and logic as language" (van Heijenhoort 1985; see also Sluga 1980). Van Heijenhoort's paper is an inspiration for much work arguing that Frege and Russell share a common conception of logic as a "maximally general science" (see the references in footnote 3 above). For van Heijenhoort, Frege and Russell are the founders of a tradition which views logic as a language rather than a calculus. This tradition is characterized by universality in at least two, related, senses: the quantifiers of the new logical language bind variables that are unrestricted in generality; and everything that can be expressed can be expressed in the new logical language, so that there is no room for a metaperspective. Opposed to the Frege-Russell tradition, van Heijenhoort sees an algebraic tradition in logic running from Boole through Schröder to Löwenheim, which views logic as a calculus, rather than a language.

Gregory Landini, while disagreeing with much of what van Heijenhoort takes to follow from this, notes that "Russell (as Frege before him) spoke of himself as offering not merely a caluclus ratiocinator in the manner of Boole, but a characteristica lingua universalis as Leibniz had conceived of it." (Landini 1998, 31, referring to CPBR3, 369.) Yet, as Hans Sluga and Volcker Peckhaus have pointed out, even Schröder, for van Heijenhoort a prime representative of the "logic as calculus" school, makes this claim for his own, Boolean, logic, arguing that it is Frege's logic that is a mere calculus. (Sluga 1987,82-3; Peckhaus 2004, 598-602) This fact suggests that we have to take be careful in estimating the significance of any claim on Russell's part to be offering a universal language. As we'll see Russell's discussions of this idea are heavily influenced by his own debt to the algebraic tradition in logic.

[16] This is the passage cited by Landini as evidence that Russell offers a Leibnizean universal characteristic.

[17] In a letter to Couturat of 17 January, 1901, Russell lauds Peano's symbolism as an "algorithm" which "I now entirely use," citing as its advantages "(1) that logical analysis is made very much easier; (2) that fallacies become much rarer; (3) that formulae and proofs become a thousand times more easy to understand." (Russell 2002, 205)